Published by Institute of Physics Publishing for SISSA

Received: *December 14, 2006*
Accepted: *January 25, 2007*
Published: *February 14, 2007*

N=31 is not IIB

Ulf Gran

Institute for Theoretical Physics, K.U. Leuven
Celestijnenlaan 200D, B-3001 Leuven, Belgium
E-mail: ulf.gran@fys.kuleuven.be

Jan Gutowski

DAMTP, Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge, CB3 0WA, U.K.
E-mail: J.B.Gutowski@damtp.cam.ac.uk

George Papadopoulos and Diederik Roest

Department of Mathematics, King's College London,
Strand, London WC2R 2LS, U.K.
E-mail: george.papadopoulos@kcl.ac.uk, diederik.roest@kcl.ac.uk

Abstract: We adapt the spinorial geometry method to investigate supergravity backgrounds with near maximal number of supersymmetries. We then apply the formalism to show that the IIB supergravity backgrounds with 31 supersymmetries preserve an additional supersymmetry and so they are maximally supersymmetric. This rules out the existence of IIB supergravity preons.

Keywords: Superstring Vacua, Extended Supersymmetry, Supergravity Models.

It has been known for some time that a priori in type II and eleven-dimensional supergravities there may exist backgrounds with any number of supersymmetries. This is because the holonomy of the supercovariant connection of these theories is a subgroup of $\mathrm{SL}(32,\mathbb{R})$ and so any $N < 32$ spinors have a non-trivial stability subgroup in the holonomy group. For a more detailed explanation see [1−3] for the M-theory and [4] for IIB. Furthermore, it was argued in [5] that the Killing spinor bundle $\mathcal{K}$ can be any subbundle of the Spin bundle and the spacetime geometry depends on the trivialization of $\mathcal{K}$. This is unlike what happens in the case of Riemannian and Lorentzian geometries [6, 7] and heterotic and type I supergravities[1] [8], where there are restrictions both on the number of Killing spinors and the Killing spinor bundle.

In this paper, we shall show that IIB backgrounds with 31 supersymmetries are maximally supersymmetric. Backgrounds with 31 supersymmetries have been considered before in the context of M-theory [9] and have been termed as preons. To our knowledge this is the first example which demonstrates that there are restrictions on the number of supersymmetries of type II backgrounds. To do this, we shall adapt the spinorial method [10] of solving Killing spinor equations to backgrounds that admit near maximal number of supersymmetries. We shall mostly focus on IIB and eleven-dimensional supergravity but most of the analysis extends to all supergravity theories.

To adapt the spinorial method to backgrounds with near maximal number of supersymmetries, we introduce a "normal" $\mathcal{K}^\perp$ to the Killing spinor bundle $\mathcal{K}$ of a supersymmetric background. The spinors of IIB supergravity are complex positive chirality Weyl spinors, so the Spin bundle is $\mathcal{S}^c_+ = \mathcal{S}_+ \otimes \mathbb{C}$, where $\mathcal{S}_+$ is the rank sixteen bundle of positive chirality Majorana-Weyl spinors. $\mathcal{S}^c_+$ may also be thought of as an associated bundle of a principal bundle with fibre $\mathrm{SL}(32,\mathbb{R})$, the holonomy group of the supercovariant connection, acting with the fundamental representation on $\mathbb{R}^{32}$. If a background admits N Killing spinors which span the fibre of the Killing spinor bundle $\mathcal{K}$, then one has the sequence

$$0 \to \mathcal{K} \to \mathcal{S}^c_+ \to \mathcal{S}^c_+/\mathcal{K} \to 0 \; . \tag{1}$$

The inclusion $i : \mathcal{K} \to \mathcal{S}^c_+$ can be locally described as

$$\epsilon_r = \sum_{i=1}^{32} f^i{}_r \eta_i \, , \quad r = 1, \ldots, N \, , \tag{2}$$

where η_p, $p = 1, \ldots, 16$, is a basis in the space of positive chirality Majorana-Weyl spinors, $\eta_{16+p} = i\eta_p$ and the coefficients f are real spacetime functions. For our notation and spinor conventions see [5]. Any N Killing spinors related by a local $\mathrm{Spin}(9,1)$ transformation give rise to the same spacetime geometry. This is because the Killing spinor equations and the field equations of IIB supergravity are Lorentz invariant. Therefore any bundles of Killing spinors and any choice of sections related by a $\mathrm{Spin}(9,1)$ gauge transformation[2] should be identified.

[1] This is provided the parallel spinors are Killing.

[2] IIB supergravity has a $\mathrm{Spin}(9,1) \times \mathrm{U}(1)$ gauge symmetry but the restriction to $\mathrm{Spin}(9,1)$ will suffice.

JHEP02(2007)044

To construct $\mathcal{K}^\perp$, first consider the dual ${}^\star\mathcal{S}^c_+$ of $\mathcal{S}^c_+$ and introduce a basis η^i, $\eta^i(\eta_j) = \delta^i{}_j$, i.e. $\eta^{16+p} = -i\eta^p$. Next consider the sections α of ${}^\star\mathcal{S}^c_+$ that annihilate the Killing spinors ϵ_r, i.e $\alpha(\epsilon) = 0$, or equivalently

$$f^i{}_r u_i = 0\,, \qquad \alpha = u_i\eta^i\,, \tag{3}$$

where u_i are real spacetime functions. Since the matrix $f = (f^i{}_r)$ has rank N, there are $32 - N$ solutions to this equation. These solutions span the sections of the co-kernel, $\mathrm{coker}\, i \subset {}^\star\mathcal{S}^c_+$ of the inclusion map $i : \mathcal{K} \to \mathcal{S}^c_+$. It is well-known that $\mathrm{Spin}(9,1)$ has an invariant inner product $B : \mathcal{S}_+ \otimes \mathcal{S}_- \to \mathbb{R}$

$$B(\epsilon,\zeta) = -B(\zeta,\epsilon) =< B(\epsilon^*), \zeta >\,, \tag{4}$$

which extends to $B : \mathcal{S}^c_+ \otimes \mathcal{S}^c_- \to \mathbb{C}$ as a bi-linear in both entries. Next consider

$$\mathcal{B}(\epsilon,\zeta) = \mathrm{Re}\, B(\epsilon,\zeta)\,, \tag{5}$$

which defines a non-degenerate pairing $\mathcal{B} : \mathcal{S}^c_+ \otimes \mathcal{S}^c_- \to \mathbb{R}$. This in turn induces a isomorphism $j : {}^\star\mathcal{S}^c_+ \to \mathcal{S}^c_-$ as $\mathcal{B}(j(\alpha),\epsilon) = \alpha(\epsilon)$. We identify the image of j, $j(\mathrm{coker}\, i) \subset \mathcal{S}^c_-$, as the "normal" bundle $\mathcal{K}^\perp$ of $\mathcal{K}$, i.e. $j(\mathrm{coker}\, i) = \mathcal{K}^\perp$. Clearly if $\alpha \in \mathrm{coker}\, i$ and $\epsilon \in \mathcal{K}$, then $\alpha(\epsilon) = 0$, and so one gets the "orthogonality" condition,

$$\mathcal{B}(j(\alpha),\epsilon) = 0\ . \tag{6}$$

Observe that $\mathcal{S}^c_+/\mathcal{K} = {}^\star\mathcal{K}^\perp$. To write this orthogonality condition in components, introduce a basis in $\mathcal{S}^c_-$, say $\theta_{i'} = -\Gamma_0\eta_i$. Then write $j(\alpha) = \nu = n^{i'}\theta_{i'}$ and the condition (6) can be written as

$$n^{i'}\mathcal{B}_{i'j}f^j{}_r = 0\,, \tag{7}$$

where $\mathcal{B}_{i'j} = \mathcal{B}(\theta_{i'},\eta_j)$.

The condition (6), or equivalently (7), leads to a correspondence between the N Killing spinors and the $32 - N$ normal directions, i.e.

$$N \longleftrightarrow 32 - N\ . \tag{8}$$

This is because instead of specifying the Killing spinors, one can determine the normal spinors. Substituting the normal spinors into these equations, one can then solve for the Killing spinors. In addition, the construction of $\mathcal{K}^\perp$ and (6) or (7) are $\mathrm{Spin}(9,1)$ covariant. Because of this, the $\mathrm{Spin}(9,1)$ gauge symmetry can be used to bring the normal spinors instead of the Killing spinors into a canonical form. In turn, this leads to a simplification in the expression for the Killing spinors which can be used to solve the Killing spinor equations for backgrounds with near maximal number of supersymmetries. We shall demonstrate this for IIB backgrounds with 31 supersymmetries. Furthermore, one may consider cases such that the sections of $\mathcal{K}^\perp$ are invariant under some non-trivial stability subgroup of $\mathrm{Spin}(9,1)$. It is clear these cases are related to (e.g. maximal and half-maximal) G-backgrounds [5, 11], where the invariance condition was imposed on the Killing spinors. The spinorial geometry techniques that we use to investigate backgrounds with N supersymmetries can be adapted to examine backgrounds with $32-N$ supersymmetries and vice-versa.

One can easily extend the construction described above to M-theory. In particular, one again has

$$0 \to \mathcal{K} \to \mathcal{S} \to \mathcal{S}/\mathcal{K} \to 0\,, \tag{9}$$

where $\mathcal{S}$ is the spin bundle associated with the Majorana representation of Spin$(10,1)$. The inclusion map $i : \mathcal{K} \to \mathcal{S}$ can be written locally as $\epsilon_r = \sum_{i=1}^{32} f^i{}_r \eta_i$, where $f^i{}_r$ are real spacetime functions and $(\eta_i, i = 1,\ldots,32)$ is a basis of Majorana spinors. As in the IIB case, we consider the the co-kernel of the inclusion map $i : \mathcal{K} \to \mathcal{S}$, $\mathrm{coker}\, i \subset {}^\star\mathcal{S}$. It is well known that $\mathcal{S}$ admits a Spin$(10,1)$ invariant inner product B which gives rise to an isomorphism $j : {}^\star\mathcal{S} \to \mathcal{S}$. As in the IIB case, we define the normal bundle of the Killing spinor bundle as $\mathcal{K}^\perp = j(\mathrm{coker}\, i)$. In this case, $\mathcal{K}^\perp$ is a subbundle of $\mathcal{S}$ and $\mathcal{S}/\mathcal{K} = \mathcal{K}^\perp$. Taking a section $\nu = n^i\eta_i$ of $\mathcal{K}^\perp$, the orthogonality condition analogous to (6) and (7) is

$$n^i B_{ij} f^j{}_r = 0\,, \tag{10}$$

where $B_{ij} = B(\eta_i, \eta_j)$. The condition (10) is Spin$(10,1)$ covariant.

As an example consider IIB backgrounds that admit 31 supersymmetries. According to the correspondence $N \leftrightarrow 32-N$, these are related to backgrounds with one supersymmetry investigated in [12, 5]. To carry out the computation, we need to find the canonical form of spinors in $\mathcal{S}^c_-$ up to Spin$(9,1)$ transformations. It is easy to deduce using an argument similar to [12] that there are three kinds of orbits of Spin$(9,1)$ in the negative chirality Weyl spinors with stability subgroups Spin$(7) \ltimes \mathbb{R}^8$, SU$(4) \ltimes \mathbb{R}^8$ and G_2. A canonical form of these spinors is

$$\begin{aligned}&\nu_1 = (n+im)(e_5 + e_{12345})\,, \quad \nu_2 = (n-\ell+im)e_5 + (n+\ell+im)e_{12345}\,,\\ &\nu_3 = n(e_5 + e_{12345}) + im(e_1 + e_{234})\,,\end{aligned} \tag{11}$$

respectively. Using the Spin$(9,1)$ gauge symmetry, we choose $\mathcal{K}^\perp$ to lie along the directions of one of the above spinors. Consider first the ν_1 case. Write the Killing spinors as

$$\epsilon_r = f^1{}_r(1+e_{1234}) + f^{17}{}_r i(1+e_{1234}) + f^k{}_r\eta_k\,, \tag{12}$$

where η_k are remaining basis elements complementary to $1+e_{1234}$ and $i(1+e_{1234})$. In what follows, we use the basis constructed from the five types of spinors in [5]. Substituting ϵ_r into (6), we get

$$f^1{}_r n - f^{17}{}_r m = 0\ . \tag{13}$$

Without loss of generality, we take $n \neq 0$. Using this, we solve for $f^1{}_r$ and substitute back into the Killing spinors to find

$$\epsilon_r = \frac{f^{17}{}_r}{n}(m+in)(1+e_{1234}) + f^k{}_r\eta_k\ . \tag{14}$$

Similarly for the normal spinors ν_2 and ν_3, we find that

$$\epsilon_r = \frac{f^{17}{}_r}{n}[(m+in)(1+e_{1234})] + \frac{f^{18}{}_r}{n}[\ell(1+e_{1234}) - n(1-e_{1234})] + f^k{}_r\eta_k \,,$$
$$\epsilon_r = \frac{f^{19}{}_r}{n}[m(1+e_{1234}) + in(e_{15}+e_{2345})] + f^k{}_r\eta_k \,, \tag{15}$$

correspondingly, where η_k are the remaining basis elements in each case. Substituting these spinors into the algebraic Killing spinor equation and using that the rank of the matrix $(f^i{}_r)$ is 31, for the $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ case one finds that

$$P_M\Gamma^M C*[(m+in)(1+e_{1234})] + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}(m+in)(1+e_{1234}) = 0\,,$$
$$P_M\Gamma^M\eta_p = 0\,, \qquad G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\eta_p = 0\,, \qquad p = 2,\dots,16\,, \tag{16}$$

and similarly

$$P_M\Gamma^M C*[(m+in)(1+e_{1234})] + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}(m+in)(1+e_{1234}) = 0\,,$$
$$P_M\Gamma^M C*[\ell(1+e_{1234}) - n(1-e_{1234})] + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}[\ell(1+e_{1234}) - n(1-e_{1234})] = 0\,,$$
$$P_M\Gamma^M C*[i(1-e_{1234})] + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}[i(1-e_{1234})] = 0\,,$$
$$P_M\Gamma^M\eta_p = 0\,, \qquad G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\eta_p = 0\,, \qquad p = 3,\dots,16\,, \tag{17}$$

and

$$P_M\Gamma^M C*[m(1+e_{1234}) + in(e_{15}+e_{2345})] + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}[m(1+e_{1234}) + in(e_{15}+e_{2345})] = 0\,,$$
$$P_M\Gamma^M C*(i(1+e_{1234}) + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}(i(1+e_{1234}) = 0\,,$$
$$P_M\Gamma^M C*(e_{15}+e_{2345}) + \frac{1}{24}G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}(e_{15}+e_{2345}) = 0\,,$$
$$P_M\Gamma^M\eta_p = 0\,, \quad G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\eta_p = 0\,, \quad p = 2,4,\dots,16\,, \tag{18}$$

for the other two cases. The factorization of P and G flux terms on η_p occurs because some of the remaining basis elements η_k come in complex conjugate pairs $(\eta_p, i\eta_p)$, where η_p are Majorana-Weyl spinors. Since the P flux term in the Killing spinor equations contains the charge conjugation matrix, $C*\eta_p = \eta_p$ and $C*(i\eta_p) = -i\eta_p$, there is a relative sign between the P and G flux terms when the algebraic Killing spinor equation is evaluated on η_p and $i\eta_p$. It now remains to solve these equations.

First, focus on the equation $P_M\Gamma^M\eta_p = 0$. Observe that in all cases, the remaining spinors η_p contain spinors which are annihilated by either Γ^- or Γ^+. In the former case, the condition $P_M\Gamma^M\eta_p = 0$ implies that only the P_- component is non-vanishing while in the latter case implies that only the component P_+ is non-vanishing. Since spinors of both types occur, $P = 0$.

Next consider the conditions on the G flux. It turns out that (16), (17) or (18) imply that $G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\epsilon = 0$ for all spinors ϵ and so $G = 0$. To see this consider the Spin(7) $\ltimes \mathbb{R}^8$ case. Setting $P = 0$ in the first condition in (16), we deduce that $G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}(1+e_{1234}) = 0$. Since the algebraic Killing spinor equations with $P = 0$ are linear over the complex numbers, we also have that $G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}i(1+e_{1234}) = 0$. This together with the remaining conditions in (16) imply that $G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\eta_i = 0$ for all the basis elements η_i. A similar argument applies to the rest of the cases. Thus we have found that the algebraic Killing spinor equations imply that $P = G = 0$. We have also verified this by an explicit computation.

Finally, if the P and G fluxes vanish, then the gravitino Killing spinor equation of IIB supergravity becomes linear over the complex numbers. This means that backgrounds with vanishing P and G fluxes always preserve an even number of supersymmetries. Thus backgrounds with 31 supersymmetries preserve an additional supersymmetry and so they are maximally supersymmetric. In particular, they are locally isometric [13] to Minkowski spacetime, $AdS_5 \times S^5$ [14] and the maximally supersymmetric plane wave [15]. As a corollary, we have shown that IIB supergravity preons do not exist.

Our proof has relied on the algebraic Killing spinor equation of IIB supergravity and so does not straightforwardly generalize to eleven-dimensional supergravity. Nevertheless, as we have seen the normal Killing spinor bundle construction generalizes to M-theory. In addition, one can show that the 31 Killing spinors of M-theory preon backgrounds take a simple form and it may be possible to solve the Killing spinor equations. We hope to report on the existence of M-theory preons in the future.

Acknowledgments

The research of D.R. is funded by the PPARC grant PPA/G/O/2002/00475 and U.G. has a postdoctoral fellowship funded by the Research Foundation K.U. Leuven.

References

[1] C. Hull, *Holonomy and symmetry in M-theory*, `hep-th/0305039`.

[2] M.J. Duff and J.T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, *Nucl. Phys.* **B 674** (2003) 217 [`hep-th/0303140`].

[3] G. Papadopoulos and D. Tsimpis, *The holonomy of the supercovariant connection and Killing spinors*, *JHEP* **07** (2003) 018 [`hep-th/0306117`].

[4] G. Papadopoulos and D. Tsimpis, *The holonomy of IIB supercovariant connection*, *Class. and Quant. Grav.* **20** (2003) L253 [`hep-th/0307127`].

[5] U. Gran, J. Gutowski and G. Papadopoulos, *The G_2 spinorial geometry of supersymmetric IIB backgrounds*, *Class. and Quant. Grav.* **23** (2006) 143 [`hep-th/0505074`];
U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, *Systematics of IIB spinorial geometry*, *Class. and Quant. Grav.* **23** (2006) 1617 [`hep-th/0507087`].

[6] M. Berger, *Sur les groupes d'holonomie homogéne des variétés á connexion affines et des variétés riemanniennes*, *Bulletin de la Société Mathématique de France* **83** (1955) 279;
McKenzie Y. Wang, *Parallel spinors and parallel forms*, *Ann. Global Anal Geom.* **7** (1989) 59.

JHEP02(2007)044

[7] J.M. Figueroa-O'Farrill, *Breaking the M-waves*, *Class. and Quant. Grav.* **17** (2000) 2925 [hep-th/9904124].

[8] U. Gran, P. Lohrmann and G. Papadopoulos, *The spinorial geometry of supersymmetric heterotic string backgrounds*, *JHEP* **02** (2006) 063 [hep-th/0510176].

[9] I.A. Bandos, J.A. de Azcarraga, J.M. Izquierdo and J. Lukierski, *BPS states in M-theory and twistorial constituents*, *Phys. Rev. Lett.* **86** (2001) 4451 [hep-th/0101113];
I.A. Bandos, J.A. de Azcarraga, J.M. Izquierdo, M. Picon and O. Varela, *On BPS preons, generalized holonomies and $D = 11$ supergravities*, *Phys. Rev.* **D 69** (2004) 105010 [hep-th/0312266].

[10] J. Gillard, U. Gran and G. Papadopoulos, *The spinorial geometry of supersymmetric backgrounds*, *Class. and Quant. Grav.* **22** (2005) 1033 [hep-th/0410155].

[11] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, *Maximally supersymmetric G-backgrounds of IIB supergravity*, *Nucl. Phys.* **B 753** (2006) 118 [hep-th/0604079].

[12] U. Gran, J. Gutowski and G. Papadopoulos, *The spinorial geometry of supersymmetric IIB backgrounds*, *Class. and Quant. Grav.* **22** (2005) 2453 [hep-th/0501177].

[13] J. Figueroa-O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten- and eleven- dimensional supergravities*, *JHEP* **03** (2003) 048 [hep-th/0211089]; *Pluecker-type relations for orthogonal planes*, math.AG/0211170.

[14] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, *Nucl. Phys.* **B 226** (1983) 269.

[15] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *JHEP* **01** (2002) 047 [hep-th/0110242].